ROCKLAND SCIENTIFIC

RSI Technical Note 030

On the Forms of the Velocity, Shear, and Rate-of-Strain Spectra

Rolf Lueck

2014-03-06

Rockland Scientific International Inc.
520 Dupplin Rd
Victoria, BC, CANADA, V8Z 1C1
www.rocklandscientific.com

Contents

1 Introduction **2**

2 Background **2**

3 The Orthogonal Case **2**

4 The Parallel Case **3**

5 The problem of wavenumber units **3**

6 The General Case **4**

7 Upper Limit of the Inertial Subrange **4**

List of Figures

1 The non-dimensional wavenumber spectrum (2) scaled by $(\hat{k}/k_s)^{-1/3}$ to make its level flat within the inertial subrange. 4

List of Tables

1 The Kolmogorov constant, A, for the cases of a velocity component that is parallel, Φ_{11}, and orthogonal, Φ_{22}, to the direction of profiling, and for the two different units of wavenumber. 4

1 Introduction

This document describes the relationship between the spectrum of velocity, estimated from data collected with an ADCP or an ADV, to the spectrum of shear, estimated from data collected with a shear probe. It also settles the confusion brought about by expressing the spectrum of velocity in the inertial subrange using units of cpm (cycles per meter) compared to units of rad m^{-1} (radians per meter).

2 Background

When the velocity and shear are measured at a fixed location, we usually use the mean flow to provide a "profile" against the direction of the flow, say the x-direction.

The spectrum of the velocity component that is orthogonal to the direction of profiling, is conventionally designated by Φ_{22}, and this is considered the "orthogonal" case. For each spatial direction there are two velocity components in the orthogonal directions, for example, $v(x)$ and $w(x)$. The gradients of these velocity components are then shears. A velocity field can have six components of shear – one pair for each of the three possible directions.

The spectrum of the velocity component that is parallel to the flow is designated by Φ_{11} and this is considered the "parallel" case. The gradient of this velocity component is the rate-of-strain. There are three such possibilities, $u(x)$, $v(y)$, and $w(z)$, where the designated velocity components are parallel to the three orthogonal directions, x, y, and z.

It is possible for velocity and shear data to be collected by literally profiling through the flow, which is frequently done using airfoil-type shear probes mounted on a freely falling instrument. The above convention applies equally to this method of collecting data.

3 The Orthogonal Case

Let w be a velocity component measured along the direction x where the direction of the velocity is orthogonal to that of x. The spectra of velocity and shear are related by

$$\left(2\pi\hat{k}\right)^2 \Phi_{22}(\hat{k}) = \Psi_S(\hat{k}) = k_s^2 \left(\epsilon\nu^5\right)^{1/4} G_2 \left(\hat{k}/k_s\right) \tag{1}$$

where G_2 is the Nasmyth shear spectrum tabulated by [1], Φ_{22} is the velocity spectrum, Ψ_S is the shear spectrum, $\hat{k}$ is the cyclic wavenumber in the x direction (in units of cpm), $k_s = (\epsilon/\nu^3)^{1/4}$ is the Kolmogorov wavenumber, ϵ the rate of dissipation of turbulence kinetic energy, and ν the kinematic viscosity.

An analytic form has been fitted to G_2 by Wolk et. al [2] and this fit has been further refined by RSI to yield

$$\Psi_S\left(\hat{k}/k_s\right) = \frac{8.05\left(\hat{k}/k_s\right)^{1/3}}{1+20.6\left(\hat{k}/k_s\right)^{3.715}} \quad . \tag{2}$$

For simplicity, I will ignore the denominator in (2), unless its inclusion is important. Therefore, the velocity spectrum in the inertial subrange is given by

$$\Phi_{22}(\hat{k}) = \frac{\alpha}{(2\pi)^2}\epsilon^{2/3}\hat{k}^{-5/3} = 0.204\ \epsilon^{2/3}\hat{k}^{-5/3} \tag{3}$$

where $\alpha = 8.05$ and 0.204 is the Kolmogorov constant for the orthogonal case, when the wavenumber is expressed in units of cpm. This value is only 2 % larger than 1/5.

4 The Parallel Case

When the velocity component is in the same direction as the profile, then the spectrum of velocity and of the rate-of-strain are both reduced by a factor of 3/4 in the inertial subrange. That is, the velocity spectrum is given by

$$\Phi_{11}(\hat{k}) = 0.153\ \epsilon^{2/3}\hat{k}^{-5/3} \quad . \tag{4}$$

The Kolmogorov constant for the parallel case is not significantly different from 3/20.

5 The problem of wavenumber units

There are many, particularly at Dalhousie University, that express the velocity spectrum in terms of a wavenumber in units of radians per meter, which results in a spectrum of identical form to (3) and (4) but with a different Kolmogorov constant. To translate from a spectrum in units of cpm to one in units of rad m^{-1} we have the requirement that

$$\Phi_{22}(k)\ \mathrm{d}k = \phi_{22}(\hat{k})\ \mathrm{d}\hat{k} \tag{5}$$

so that both spectra integrate to give the same variance of w, where $k = 2\pi\hat{k}$, is the wavenumber in units of rad m^{-1}. Direct substitution into (3) gives

$$\Phi_{22}(k)\ dk = \frac{\alpha}{(2\pi)^2}\epsilon^{2/3}\left(\frac{k}{2\pi}\right)^{-5/3}\frac{\mathrm{d}k}{2\pi} \quad , \tag{6}$$

which reduces to

$$\Phi_{22}(k) = 0.694\ \epsilon^{2/3}k^{-5/3} \quad . \tag{7}$$

Thus, the Kolmogorov constant based on the Nasmyth spectrum, when it is expressed in units of rad m^{-1}, is 0.693 which is 7 % larger than the value of 0.65 used by Alex Hay in his OS poster shown in Honolulu in 2014. These values are not significantly different from 2/3 because there are probably no high Reynolds number measurements accurate to better than 4 %. The Kolmogorov constant for units of rad m^{-1} is a factor of $(2\pi)^{2/3}$ larger than the constant for a spectrum in units of cpm.

6 The General Case

Let A be the Kolmogorov constant regardless of units used for the wavenumber. The rate of dissipation can then be deciphered directly by plotting

$$\epsilon = \left(\frac{1}{A}\Phi_{ii}(k)k^{5/3}\right)^{3/2} \tag{8}$$

where the value of A depends on the units of wavenumber k, the velocity component, and the direction of profiling. The values of $i = 1$ and $i = 2$ represent the parallel and orthogonal cases, respectively. The Kolmogorov constant is summarized in Table 1.

Direction	A [rad m^{-1}]	A [cpm]
Φ_{11}	1/2	3/20
Φ_{22}	2/3	1/5

Table 1: The Kolmogorov constant, A, for the cases of a velocity component that is parallel, Φ_{11}, and orthogonal, Φ_{22}, to the direction of profiling, and for the two different units of wavenumber.

7 Upper Limit of the Inertial Subrange

At very high wavenumbers viscosity dampens the turbulent velocity fluctuations, and this is indicated by the numerator in the non-dimensional shear spectrum (2). One way to quantify the onset of viscous damping is to plot the non-dimensional spectrum multiplied by the wavenumber raised to the power of $-1/3$ (Figure 1). This makes the spectrum flat within

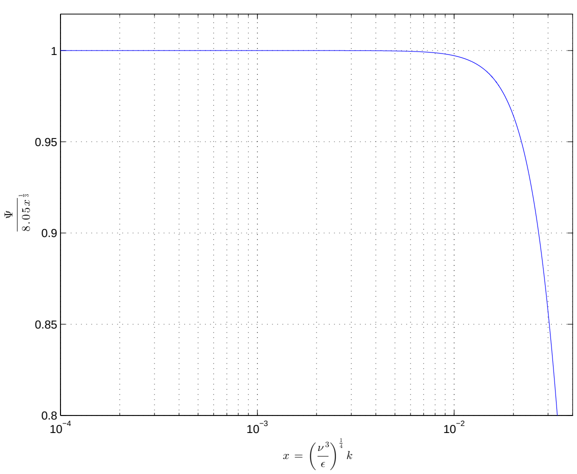

Figure 1: The non-dimensional wavenumber spectrum (2) scaled by $(\hat{k}/k_s)^{-1/3}$ to make its level flat within the inertial subrange.

the inertial subrange. The spectrum falls below the power law of $-5/3$ by 10 % at a non-dimensional wavenumber of $\hat{k}/k_s = 0.027$, which I somewhat arbitrarily will define as the upper limit of the inertial subrange. If the wavenumber is expressed in units of rad m^{-1} then the upper limit is larger by 2π and is at $k/k_s = 0.16$. This applies to the orthogonal case.

For the parallel case, the level of the velocity spectrum is lower by a factor of 3/4 and, consequently, the inertial subrange must be larger by approximately a factor of 4/3.

References

[1] N. S. Oakey. Determination of the rate of dissipation of turbulent kinetic energy from simultaneous temperature and velocity shear microstructure measurements. *Journal of Physical Oceanography*, 12:256–271, 1982.

[2] F. Wolk, H. Yamazaki, L. Seuront, and R. G. Lueck. A new free-fall profiler for measuring bio-physical microstructure. *Journal of Atmospheric and Oceanic Technology*, 19:780–793, 2002.

End Of Document